8.1 – General Linear Transformations

Definition: (analogous to Theorem 1.8.2)
If $T : V \to W$ is a mapping from a vector space V to a vector space W, then T is called a **linear transformation** from V to W if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars k:
a) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)
b) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V.

using $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$

Theorem 8.1.1 (analogous to Theorem 1.8.1)
If $T : V \to W$ is a linear transformation, then:
a) $T(\mathbf{0}) = \mathbf{0}$.
b) $T(\mathbf{u}-\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.
c) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

$T(\vec{u}) = T(\vec{u}+\vec{0}) = T(\vec{u})+T(\vec{0})$
$\Rightarrow T(\vec{0}) = \vec{0}$

pf: a) Let $k=0$
$T(\vec{0}) = T(0\vec{u}) = 0T(\vec{u}) = \vec{0}$

Suppose $T(\vec{0}) \neq \vec{0}$,
Then $T(\vec{a}-\vec{a}) \neq \vec{0}$
$\Rightarrow T(\vec{a}) - T(\vec{a}) \neq \vec{0}$
$\Rightarrow T(\vec{a}) \neq T(\vec{a})$

Examples of Linear transformations

- $T : V \to W$ defined by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (the zero transformation)
- $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ (the identity operator)
- $T : V \to V$ defined by $T(\mathbf{x}) = c\mathbf{x}$ (contraction if $0 < c < 1$ and dilation if $c > 1$)
- $T : V \to R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ (inner product of $\mathbf{x}$ with $\mathbf{x}_0$) [We'll see this in Ch. 6]
- $T : M_{nn} \to M_{nn}$ defined by $T(A) = A^T$

x	$y=f(x)$
x_1	$f(x_1)$
x_2	$f(x_2)$

x_3 $f(x_3)$

- $T : V \to R^n$ defined by $T(f) = (f(x_1),\ f(x_2),\ \ldots, f(x_n))$, where V is a subspace of $F(-\infty,\ \infty)$, and $x_1, x_2, \ldots, x_n$ is a sequence of real numbers (evaluation transformation)
- $D : C^1(-\infty, \infty) \to F(-\infty,\ \infty)$ defined by $D(\mathbf{f}) = f'(x)$ (differentiation)
- $J : C(-\infty, \infty) \to C^1(-\infty, \infty)$ defined by $J(\mathbf{f}) = \int_0^x f(t)\, dt$ (integration)

Theorem 8.1.2 Let $T : V \to W$ be a linear transformation, for which the vector space V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v}$ in V can be expressed as
$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n)$ where $c_1, c_2, \ldots,\ c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination of the vectors in the basis S.

That is, linear transformations preserve linear combos.

Pf: Let $\vec{v} \in V$. Then there are $c_i \in \mathbb{R}$ $\ni$

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i \Rightarrow T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$$

$$= \sum_{i=1}^{n} T(c_i\vec{v}_i) = \sum_{i=1}^{n} c_i T(\vec{v}_i) \checkmark$$

#20 Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2,\ 1)$ and $\mathbf{v}_2 = (1,\ 3)$, and let $T : R^2 \to R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1,\ 2,\ 0)$ and $T(\mathbf{v}_2) = (0,\ -3,\ 5)$. Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2,\ -3)$.

When we did this in 1.8, we wanted the standard matrix for the transformation. Here, we'll find the formula.

We start by expressing $\vec{x} = (x_1, x_2)$ as a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

Road map: $\vec{x} = a\vec{v}_1 + b\vec{v}_2 \Rightarrow T(\vec{x}) = aT(\vec{v}_1) + bT(\vec{v}_2)$

Note: $(a, b) = (\vec{v})_S$

$$\left[\begin{array}{cc|c} -2 & 1 & x_1 \\ 1 & 3 & x_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{7}x_1 + \frac{1}{7}x_2 \\ 0 & 1 & \frac{1}{7}x_1 + \frac{2}{7}x_2 \end{array}\right]$$

$$(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)\underline{(-2, 1)} + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)\underline{(1, 3)}$$

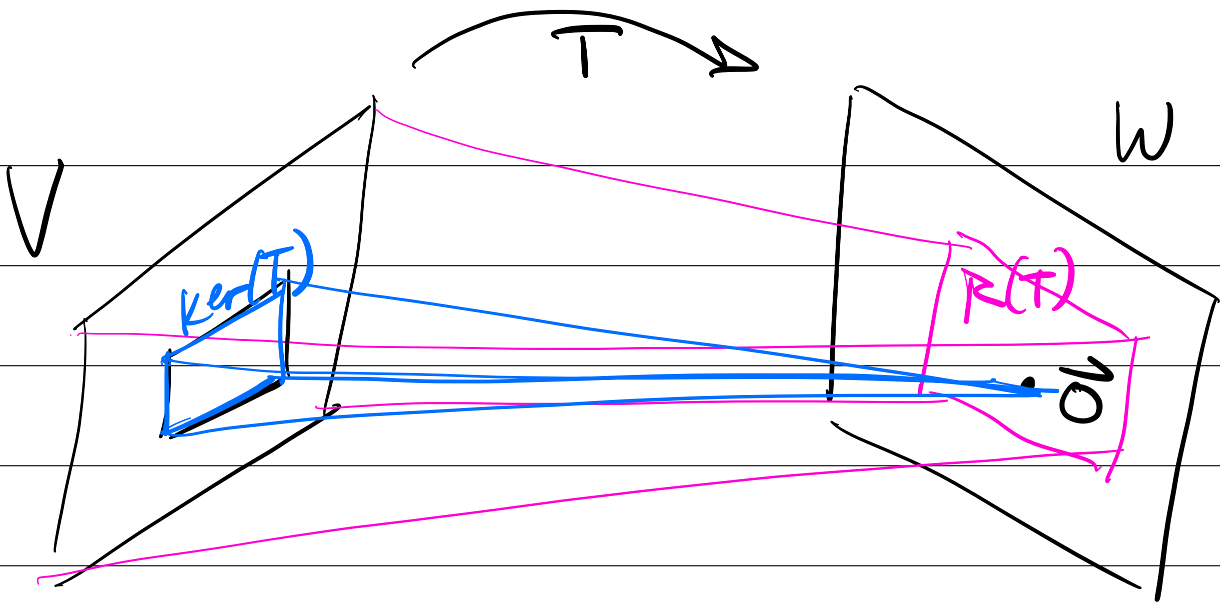
$$\Rightarrow T(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1, 2, 0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0, -3, 5)$$

$$= \left(\underline{\frac{3}{7}x_1 - \frac{1}{7}x_2}, \underline{-\frac{6}{7}x_1 + \frac{2}{7}x_2}, \underline{0}\right) + \left(\underline{0}, \underline{-\frac{3}{7}x_1 - \frac{6}{7}x_2}, \underline{\frac{5}{7}x_1 + \frac{10}{7}x_2}\right)$$

$$\boxed{T(x_1, x_2) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)}$$

$$T(2, -3) = \left(\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7}\right)$$

Definition: (analogous to definitions seen in Section 4.2 (**kernel**) and Section 1.8 (**range**); also related to Definition 2 of Section 4.8 (**null space**) and, by Theorem 4.8.1, column space)
If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.



#7 Determine whether the mapping T is a linear transformation, and if so, find its kernel.
$T : P_2 \to P_2$, where
a. $T\left(a_0 + a_1x + a_2x^2\right) = a_0 + a_1(x+1) + a_2(x+1)^2$
b. $T\left(a_0 + a_1x + a_2x^2\right) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$

a) $T(\vec{0}) = \vec{0}$ maybe it's linear.
Let $\vec{p} = p_0 + p_1x + p_2x^2$, $\vec{q} = q_0 + q_1x + q_2x^2$, $k \in \mathbb{R}$
$T(k\vec{p} + \vec{q}) = T(k(p_0 + p_1x + p_2x^2) + q_0 + q_1x + q_2x^2)$
$= T(kp_0 + q_0 + (kp_1 + q_1)x + (kp_2 + q_2)x^2)$

$= kp_0+q_0+(kp_1+q_1)(x+1)+(kp_2+q_2)(x+1)^2$

$= kp_0+kp_1(x+1)+kp_2(x+1)^2+q_0+q_1(x+1)+q_2(x+1)^2$

$= k\left[p_0+p_1(x+1)+p_2(x+1)^2\right]+q_0+q_1(x+1)+q_2(x+1)^2$

$= kT(\vec{p})+T(\vec{q})$ ✓

b) $T(\vec{0})=1+x+x^2 \neq \vec{0}$ not linear.

#10 Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

a. x^2 $\quad T(x^2)=x^3$ ×

b. 0

c. $1+x$ $\quad T(1+x)=x(1+x)$ ×

d. $-x$

$T(-x)=-x^2$ ×

$\ker(T)=\{p(x)\in P_2 \mid xp(x)=0 \ \forall x\}$

$\ker(T)=\{0\}$

#11 Let $T : P_2 \to P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$? $\quad T(p(x))=xp(x)$

a. $x+x^2$

~~b.~~ $1+x$

~~c.~~ $3-x^2$

d. $-x$

Since $T(a_0+a_1x+a_2x^2)=a_0x+a_1x^2+a_2x^3$,

the constant term is 0.

a, d $\in R(T)$.

#25 Let $T_A : R^4 \to R^3$ be multiplication by A. Find a basis for the kernel of T_A, and then find a basis for the range of T_A that consists of column vectors of A.

col(A)

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} x_4$$

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= \tfrac{10}{7} x_4 \\ x_2 &= \tfrac{2}{7} x_4 \\ x_3 &= 0 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for kernel is $\left\{ \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$.

pivot columns of R point to columns of A that form a basis for col(A), which is $R(T_A)$.

Basis: $\left\{ \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

Theorem 8.1.3 If $T : V \to W$ is a linear transformation, then:
a) The kernel of T is a subspace of V.
b) The range of T is a subspace of W.

Definition: (analogous to Definition 1 of Section 4.9)
Let $T : V \to W$ be a linear transformation. In the case that the range of T is finite-dimensional its dimension is called the **rank of T**, denoted by rank(T); and if the kernel of T is finite-dimensional, then its dimension is called the **nullity of T**, denoted by nullity(T).

Theorem 8.1.4 Dimension Theorem for Linear Transformations (generalization of Theorem 4.9.2)

If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and
rank(T) + nullity(T) = dim(V).

rank + nullity = dim of domain

#13 In each part, use the given information to find the nullity of the linear transformation T.

a. $T : R^5 \to P_5$ has rank 3.
b. $T : P_4 \to P_3$ has rank 1.
c. The range of $T : M_{mn} \to R^3$ is R^3. — rank = 3
d. $T : M_{22} \to M_{22}$ has rank 3.

a) Dim of $R^5 = 5$ so $5-3=2$ = nullity(T)

b) Dim of $P_4 = 5$ so $5-1=4$ = nullity(T)

c) nullity$(T) = mn - 3$

d) Dim of $M_{22} = 4$ so $4-3=1$ = nullity(T)